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Critical behaviour of the discrete spin cubic model†

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Abstract. An n-component spin model, with the nearest-neighbour Hamiltonian

$$\mathscr{H}=-J\sum_{\langle ij\rangle}(\boldsymbol{S}_i\,.\,\boldsymbol{S}_j)-K\sum_{\langle ij\rangle}(\boldsymbol{S}_i\,.\,\boldsymbol{S}_j)^2,$$

where $\boldsymbol{S}_i$ is a discrete unit vector pointing only along one of the $2n$ cubic axes directions, is studied exactly at dimensions $d=1$ and $d=1+\epsilon$ and approximately (using dedecoration renormalization group recursion relations) at $d=2$. The model exhibits four competing possible types of critical behaviour, related to the Ising model, to the n-state and $2n$-state Potts models and to a 'new cubic' fixed point. For large n, at $d=2$, the last three types of behaviour show peculiarities which may be related to the transition becoming a first-order one.

1. Introduction

Cubic symmetry plays an important role in determining the nature of the displacive phase transitions of perovskites (Bruce and Aharony 1975, Aharony and Bruce 1974). Similarly, such symmetry may be relevant for some ferromagnets (Aharony and Bruce 1975). Therefore, cubic systems have recently been thoroughly investigated using the renormalization group technique at $d=4-\epsilon$ dimensions (Aharony 1973, Bruce 1974, Natterman and Trimper 1975, Ketley and Wallace 1973) and at a large number of components of the order parameter (Wallace 1973).

In these studies, one usually considers a continuous n-component spin model, with the effective Hamiltonian

$$-\beta\mathscr{H}=\int \mathrm{d}^dx\Big(\tfrac{1}{2}[r|\boldsymbol{S}(\boldsymbol{x})|^2+(\boldsymbol{\nabla}\boldsymbol{S})^2]+u|\boldsymbol{S}|^4+v\sum_{\alpha=1}^{n}(S_\alpha)^4+\dots\Big), \tag{1}$$

where $\boldsymbol{S}$ is a continuous n-component 'spin' order parameter. The parameter v measures the strength of the cubic anisotropy. In mean field theory, the order parameter $\boldsymbol{S}$ will be along a cubic diagonal if $v>0$ and along a cubic axis if $v<0$. These directions will be maintained when fluctuations are present, as long as T_c is not reached (Bruce and Aharony 1975).

Renormalization group studies at $d=4-\epsilon$ (Aharony 1973) yield four fixed points with u and v of order ϵ. These are Gaussian, with $u^*=v^*=0$, isotropic 'Heisenberg', with $v^*=0$, Ising, with $u^*=0$ (n decoupled Ising models), and 'cubic', with both u^* and v^* non-zero. This 'cubic' fixed point is the most stable (with $v^*>0$) for $n>n_c(d)$ and becomes unstable (with $v^*<0$) for $n<n_c(d)$. The value of $n_c(d)$ approaches 2 as

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$d \to 2^+$ (Pelcovits and Nelson 1976, Brézin *et al* 1976) and 4 as $d \to 4^-$ (Aharony 1973, Ketley and Wallace 1973). Combining the expansions above $d = 2$ and below $d = 4$ one finds that $n_c(3)$ is probably larger than 3 and smaller than 4. For $d \leq 2$, there is no ferromagnetic long-range order at finite temperature for the isotropic case $n \geq 2$ (Mermin and Wagner 1966, Hohenberg 1967). This theorem does not apply to the cubic case. One may thus expect some kind of long-range order for the cubic problem even at dimensions $1 < d \leq 2$.

To decide which of all the possible fixed points describes the critical behaviour of a given system, one must iterate the renormalization group recursion relations, starting with the Hamiltonian of that system, and follow the resulting changes in the effective Hamiltonian (or 'flow diagrams', in Hamiltonian space), until a fixed point is reached (Wilson and Kogut 1974). The flow diagrams of the Hamiltonian (1) in the u–v plane under renormalization group recursion relations near $d = 4$ are described in detail elsewhere (Aharony 1976). Generally, if $v > 0$ then the Hamiltonian 'flows' to the stable cubic $(n > n_c)$ or Heisenberg $(n < n_c)$ fixed point (we assume that $u > 0$). However, the situation for $v < 0$ is more complicated. If $n > n_c$, v flows towards $-\infty$, thus leaving the range of applicability of the perturbation expansion. If $n < n_c$ then the Hamiltonian flows to the Heisenberg fixed point for small $|v|$, and towards $-\infty$ for values of $|v|$ which exceed some 'tricritical' value (Aharony 1974†). These flows towards $-\infty$, termed 'runaways', are usually interpreted as corresponding to the transition becoming a first-order one. This is based on the fact that the flow eventually reaches a region in which the quartic terms in (1) are not positive definite and in which mean field theory can be used. Similar conclusions were found by Wallace (1973), for small $v < 0$ and large n, for dimensionalities $2 < d < 4$, and by Rudnick (unpublished).

Since all these studies were limited to small values of v (of order ϵ), due to their diagrammatic nature, it is highly desirable to consider the case of large negative v using some independent technique. A model which corresponds to this limit was recently introduced by Levy and co-workers (Kim *et al* 1975, Kim and Levy 1975, Kim *et al* 1976). The model, which they called 'the n-component cubic model', is based on the Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \delta_{\alpha_i \alpha_j}, \tag{2}$$

where σ_i takes the values ± 1 and α_i the values $1, 2, \ldots, n$. This Hamiltonian is the same as

$$\mathcal{H} = -J \sum_{\langle ij \rangle} (\boldsymbol{S}_i \cdot \boldsymbol{S}_j), \tag{3}$$

where $\boldsymbol{S}_i$ can assume the $2n$ values $(\pm 1, 0, \ldots, 0)$, $(0, \pm 1, 0, \ldots), \ldots, (0, \ldots, 0, \pm 1)$. Clearly (3) is equivalent to the $v \to -\infty$ limit of

$$\mathcal{H} = -J \sum_{\langle ij \rangle} (\boldsymbol{S}_i \cdot \boldsymbol{S}_j) + v \sum_i \sum_{\alpha=1}^{n} (S_{i_\alpha})^4, \tag{4}$$

where $\boldsymbol{S}_i$ is a classical spin vector (on the unit sphere), or to the limit $v \to -\infty$, $u \to \infty$ of (1) (Wilson and Kogut 1974). Levy and co-workers show that it is also applicable to the description of phase transitions in cubic rare earths which have sixfold-degenerate ground states. Kim and Levy (1975) used high-temperature series expansions of the

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model (2) for FCC lattices to conclude that it has a first-order transition for $n > n^*$, with $n^* \simeq 2{\cdot}35 \pm 0{\cdot}2$, and a continuous transition for $n < n^*$. Note that for $n = 2$, equation (2) is equivalent to two decoupled Ising models and thus has a second-order transition.

In this paper we study various aspects of the critical behaviour of an extension of the cubic Hamiltonian, equation (2), at low dimensionalities. The model is described in detail, including various exact limits, in § 2. In particular, the extended model reduces to versions of the Potts (1952) model in several limits. Then § 3 is devoted to an exact solution of the model at $d = 1$. Although it has no long-range order at finite temperatures, it approaches criticality as $T \to 0$ and one can get some feeling of its critical properties in this limit. In §4 we extend these results to $d = 1 + \epsilon$ dimensions, following the technique of Migdal (1975), and in § 5 we discuss the model at $d = 2$, using a simple approximate dedecoration renormalization group (Barber 1975). Finally, the results are discussed and summarized in § 6.

2. The model

In a similar way to equation (3), one can also consider a quadrupolar interaction (Kim *et al* 1975)

$$\mathscr{H}_{\mathrm{q}} = -K \sum_{\langle ij \rangle} (\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^2. \tag{5}$$

In the notation of equation (2), $(\boldsymbol{S}_i \cdot \boldsymbol{S}_j) = \sigma_i \sigma_j \delta_{\alpha_i \alpha_j}$, and therefore

$$\mathscr{H}_{\mathrm{q}} = -K \sum_{\langle ij \rangle} \delta_{\alpha_i \alpha_j} \tag{6}$$

The model we shall consider is the sum of equations (2) and (6), i.e.

$$\mathscr{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \delta_{\alpha_i \alpha_j} - K \sum_{\langle ij \rangle} \delta_{\alpha_i \alpha_j}. \tag{7}$$

A direct motivation for studying (7) arises from the fact that $\mathscr{H}_{\mathrm{q}}$ is generated under the renormalization group transformation even if we start only with the Hamiltonian (2). However, the Hamiltonian (7) is physically relevant to cubic systems with both dipolar and quadrupolar (short-range) coupling, and has many additional interesting limiting cases. Clearly, it goes to equation (1) in the continuous spin limit. In the remainder of this section we shall review some of its limiting cases.

First, note that (6) is just the Hamiltonian of the n-state Potts model (Potts 1952). It reduces to the Ising model for $n = 2$, and is directly related to the percolation problem if analytically continued to $n \to 1$ (Kasteleyn and Fortuin 1969, Fortuin and Kasteleyn 1972, Harris *et al* 1975). The limit $n \to 0$ also corresponds to a percolation problem (Stephen 1976). Harris *et al* (1975) studied the model using the Niemeijer and van Leeuwen (1974) renormalization group at $d = 2$ for $n \to 1$, and Stephen (1976) studied it using the Migdal (1975) renormalization group at $d = 1 + \epsilon$. Both authors studied it also at $d = 6 - \epsilon$, as was also done by Priest and Lubensky (1976). Low- and high-temperature series also exist, both at $d = 2$ and at $d = 3$, for various values of n (for recent references, see Enting and Domb 1975, Kim and Joseph 1975). These seem to yield a second-order transition for small values of n, and a first-order transition for large values of n. Baxter (1973) recently proved, that at $d = 2$ the transition becomes first-order for $n > 4$. Mean field theory predicts a first-order transition for all $n > 2$ (Straley and Fisher 1973).

The Hamiltonian (2) can also be written in the form

$$\mathcal{H} = -\tilde{J} \sum_{\langle ij \rangle} \tilde{\sigma}_i \tilde{\sigma}_j \delta_{\alpha_i \alpha_j}, \tag{8}$$

with $\tilde{J} = J/n$ and $\tilde{\sigma}_i = \pm\sqrt{n}$. In the limit $n \to 0$, this Hamiltonian corresponds to the self-avoiding walk problem. This was recently shown by Hilhorst (1976), who also applied to (8) a special form of the Niemeijer and van Leeuwen (1974) renormalization group transformation on a triangular lattice.

We now turn to study some special cases of our combined Hamiltonian (7). For $n = 1$, it trivially reduces to the Ising model. For $n = 2$ we can write

$$\delta_{\alpha_i \alpha_j} = (1 + \tau_i \tau_j \sigma_i \sigma_j)/2, \tag{9}$$

with $\tau_i = \pm 1$. Thus, (7) becomes

$$\mathcal{H} = -\tfrac{1}{2} J \sum_{\langle ij \rangle} (\sigma_i \sigma_j + \tau_i \tau_j) - \tfrac{1}{2} K \sum_{\langle ij \rangle} \sigma_i \sigma_j \tau_i \tau_j - \tfrac{1}{2} KN. \tag{10}$$

This is exactly of the form of the Ashkin and Teller (1943) model; for $K = 0$ we have two decoupled Ising models, with exchange $\frac{1}{2}J$, while non-zero values of K introduce coupling of the energy–energy type between them (see also Kadanoff and Wegner 1971). At $d = 2$, this model should yield exponents which vary continuously with K. A renormalization group study of the $n = 2$ case was recently done by Knops (1975).

Another special case is found if $K = J$. Using $2\delta_{\sigma_i \sigma_j} \equiv 1 + \sigma_i \sigma_j$, equation (7) becomes

$$\mathcal{H} = -2J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j} \delta_{\alpha_i \alpha_j}, \tag{11}$$

which corresponds to the $2n$-state Potts model, with coupling constant $2J$.

Finally, we note that when $K \to \infty$, all sites prefer to have the same value of α_j, and thus the problem reduces to an Ising model along one of the axes.

3. Solution in one dimension

To solve the problem exactly in one dimension, it is convenient to find the eigenvalues of the $2n \times 2n$ transfer matrix,

$$\sum_{\sigma_2 = \pm 1} \sum_{\alpha_2 = 1}^{n} e^{\beta(J\sigma_1\sigma_2 + K)\delta_{\alpha_1\alpha_2}} \psi(\sigma_2, \alpha_2) = \lambda \psi(\sigma_1, \alpha_1), \tag{12}$$

where $\beta = 1/k_B T$. This equation has three eigenvalues,

$$\begin{aligned} &\lambda_1 = 2(B+n) &&\text{(non-degenerate)},\\ &\lambda_2 = 2B &&\text{((}n-1\text{)-fold degenerate)},\\ &\lambda_3 = 2A &&\text{(}n\text{-fold degenerate)}, \end{aligned} \tag{13}$$

where

$$A = e^{\beta K} \sinh \beta J, \qquad B = e^{\beta K} \cosh \beta J - 1. \tag{14}$$

Thus, the partition function for a chain of N particles is (Domb 1960):

$$Z = \lambda_1^N + (n-1)\lambda_2^N + n\lambda_3^N. \tag{15}$$

For $n>0$, one has $\lambda_1>\lambda_2$ at all $T>0$. For $n>1$ one also has $\lambda_1>\lambda_3$. Thus, the free energy per spin in the thermodynamic limit ($N\to\infty$) becomes

$$F=-k_BT\ln\lambda_1=-k_BT\ln[2(e^{\beta K}\cosh\beta J+n-1)], \tag{16}$$

for $n>1$, $T>0$. As $T\to\bar{0}$, all eigenvalues become degenerate, provided $K+|J|>0$. In this case, $\lambda_1\simeq\lambda_2\simeq|\lambda_3|\simeq\exp[\beta(K+|J|)]\to\infty$. Thus, we can have a zero-temperature critical point with $J=0$ (n-state Potts model), $K=0$ ('new cubic'), $J=K$ ($2n$-state Potts model), etc.

To find the spin–spin correlation function, it is convenient to write the transfer operator in the form

$$e^{\beta(J\sigma_1\sigma_2+K)\delta_{\alpha_1\alpha_2}}\equiv\tfrac{1}{2}\{(\lambda_1/n)+\lambda_2[\delta_{\alpha_1\alpha_2}-(1/n)]+\lambda_3\sigma_1\sigma_2\delta_{\alpha_1\alpha_2}\}. \tag{17}$$

The operators $1/2n$, $[\delta_{\alpha_1\alpha_2}-(1/n)]/2$ and $\sigma_1\sigma_2\delta_{\alpha_1\alpha_2}/2$ are orthonormal under the trace operation, and thus one easily finds that in the thermodynamic limit

$$\langle \boldsymbol{S}_i\cdot\boldsymbol{S}_j\rangle=\langle\sigma_i\sigma_j\delta_{\alpha_i\alpha_j}\rangle=(\lambda_3/\lambda_1)^{|j-i|}, \tag{18}$$

and the correlation length ξ may be defined as

$$\xi=1/\ln(\lambda_1/\lambda_3). \tag{19}$$

A second correlation length may be defined to describe the Potts-like, or quadrupolar, correlation function $\langle\delta_{\alpha_i\alpha_j}-(1/n)\rangle$. This will be related to (λ_1/λ_2). At $T=0$, both correlation lengths diverge to infinity.

It is interesting to note, that for $0<n<1$ one may have $\lambda_1=\lambda_3$, or $\xi\to\infty$, at the *finite temperature*

$$k_BT_c=(J-K)/|\ln(1-n)|, \tag{20}$$

provided $K<J$. A similar result was found for isotropic Heisenberg-like systems with $n<1$ (Balian and Toulouse 1974). We shall not discuss this peculiar case any longer here.

For small T, a small variable is (assuming $J>0$)

$$1/\xi\simeq 2\,e^{-2\beta J}+2(n-1)\,e^{-\beta(J+K)}. \tag{21}$$

Substituting in (16) we can thus identify the critical exponents, expressed in terms of the temperature variable $1/\xi$. These turn out to be the same as for the Ising model (see e.g. Nelson and Fisher 1975), i.e. $\alpha=\nu=\eta=1$.

Equation (17) is also very useful for constructing the dedecoration renormalization group recursion relations for this problem. Eliminating $(b-1)$ out of every b spins, these recursion relations are simply

$$\lambda_i\to\lambda_i'=\lambda_i^b,\qquad i=1,2,3. \tag{22}$$

The fixed point values are $\lambda_i^*=0, 1$, and the eigenvalues $(\partial\ln\lambda_i'/\partial\ln\lambda_i)_*=b^1$ immediately yield the critical exponents mentioned above.

4. Expansion in $\epsilon=(d-1)$

The recursion relations (22) are easily generalized to $d=1+\epsilon$ dimensions, following the work of Migdal (1975). Equation (22) now becomes

$$\lambda_i((\beta J)'b^{1-d}, (\beta K)'b^{1-d})=\lambda_i(\beta J, \beta K)^b. \tag{23}$$

Dividing the equations with $i=2,3$ by that with $i=1$, we thus have two equations for the fixed point values $(\beta J)^*$ and $(\beta K)^*$. Assuming both $(\beta J)^*$ and $(\beta K)^*$ are large, these two equations immediately give

$$(\beta J)^* = (\beta K)^* = \frac{\ln \lambda}{2(1-\lambda^{1-d})} \simeq \frac{1}{\epsilon}. \tag{24}$$

Remembering equation (11), we note that this is exactly the $2n$-state Potts model fixed point. Indeed, the result (24) agrees with that of Stephen (1976) for this case.

If $(\beta J)^* = 0$ we find the n-state Potts model fixed point, with $(\beta K)^* = 1/\epsilon$. If $(\beta K)^*$ is small (of order unity compared to $1/\epsilon$) then we find a 'new cubic' fixed point, with $(\beta J)^* = 1/\epsilon$.

Linearizing the recursion relation about these fixed points we now find that the leading exponent is always ϵ, and thus

$$\nu \simeq 1/\epsilon. \tag{25}$$

The $2n$-state Potts model fixed point (24) has another eigenvalue equal to ϵ, and is thus unstable to perturbations in the J–K plane. The other two fixed points are stable, except for the temperature instability.

We thus conclude that the cubic model has a second-order phase transition for $d > 1$, with exponents which are the same (to leading order in ϵ) as those of the Ising model (Migdal 1975) or the Potts model (Stephen 1976).

5. Approximate recursion relations at $d = 2$

Since the cubic model is now expressed in terms of discrete spins, we can treat it using any one of the many existing approximate renormalization group schemes at $d = 2$. All are approximate, and not much is yet known about their convergence properties. This being the case, we shall not aim in the following discussion at an accurate numerical solution of the problem, but rather at a qualitative understanding of the fixed point structure and of the possible crossover phenomena involved. For this, we use a very simple dedecoration renormalization group, recently proposed by Barber (1975). Although it fails to give an accurate exponent η, we feel that some of the qualitative results obtained are quite general.

We thus consider a square lattice, and perform the trace in the partition function over the spins on one half of the sites, sitting on one square sub-lattice. In practice, we use a cluster approximation, in which only four spins are kept (S1 in figure 2 of Barber's paper). The trace over two of the spins is easily performed, and we end up with the recursion relations

$$\begin{aligned} A' &= 2A^2(n+2B+B^2)/(n+2B)^2, \\ B' &= [A^4+B^4+2B^2(n+2B)]/(n+2B)^2, \end{aligned} \tag{26}$$

where A and B were defined in equation (14).

One can now solve equations (26) for all the possible fixed points. These, and the appropriate flow lines, are shown (for $n = 3$) in figure 1. The structure is quite similar to that found by Knops (1975), for $n = 2$, using the Niemeijer and van Leeuwen (1974) technique, or to the results in $1+\epsilon$ dimensions. There are three fixed points at finite values of βJ and βK:

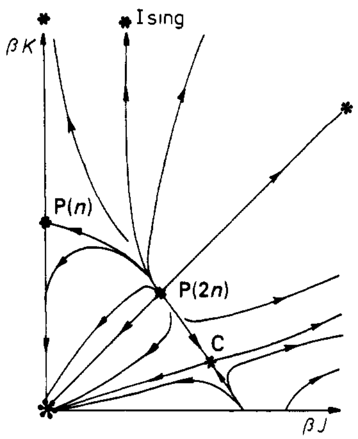

Figure 1. Fixed points and flow lines for $n = 3$ ($n < n_x$). P(n) is the n-state Potts model fixed point and C is the 'new cubic' fixed point.

P(n). $(\beta J)^* = 0$, $(\beta K)^* = \beta K_P(n)$. This fixed point describes the critical behaviour of the n-state Potts model (equation (6)).

P($2n$). $(\beta J)^* = (\beta K)^* = \frac{1}{2}\beta K_P(2n)$. This corresponds to the $2n$-state Potts model (equation (11)).

C. $(\beta J)^*$ and $(\beta K)^*$ are both finite for $n \neq 2$ ($(\beta K)^* = 0$ for $n = 2$, corresponding to two decoupled Ising models, equation (10)). This is the 'new cubic' fixed point.

In addition to these, we note that for $\beta K = \infty$, the recursion relations in βJ reduce to Barber's recursion relations for the Ising model, yielding $(\beta J)^* \simeq 0$, 0·61 or ∞. The other fixed points ($(\beta K)^* = (\beta J)^* = 0$, or $T = \infty$, and $(\beta J)^* = \infty$, or $T = 0$) have trivial interpretations.

The flow lines in figure 1 already indicate the stability of the fixed points. The n-state Potts model fixed point P(n) has one relevant variable (we consider only even spin operators, i.e. zero magnetic fields), which we relate to the temperature. This fixed point will therefore describe the physics of the phase transition if J is not too large. In fact, for $K > J$ we shall have either this behaviour or the Ising behaviour characterized by the appropriate fixed point at $(\beta K)^* = \infty$, $(\beta J)^* = (\beta J)^*_I (\simeq 0{\cdot}61)$.

The picture given in figure 1 for $n = 3$ is found to apply for a range of values of n, i.e. $0 \leq n < n_x$. Within our crude approximation, $n_x = 8$. In this range the $2n$-state Potts model fixed point P($2n$) is found to be doubly unstable, and in fact is a branch point on the critical surface (Knops 1975); two critical lines (towards P(n) and towards 'Ising') leave it for $K > J$. The 'new cubic' fixed point has only one (temperature) instability, and therefore describes the physics of the phase transition for the range $J > K$. This fixed point, in the limit $n \to 0$, will also describe the critical behaviour of the self-avoiding walk problem (Hilhorst 1976); in this case, we find $(\beta J)^*_c \simeq n/2$ and $(\beta K)^*_c \simeq -n^2/16$, so that $(\beta J)^*/n$ has a finite limit (see equation (8)). Note that $(\beta K)^*_c = 0$ for $n = 2$, and $(\beta K)^*_c < 0$ for $n < 2$.

As n increases towards n_x, the fixed point C moves towards the fixed point P($2n$), with which it coincides at $n = n_x$. For $n > n_x$, the two fixed points interchange in their roles; the 'new cubic' fixed point C now becomes doubly unstable, and plays the role of the branch point, while the $2n$-state Potts model fixed point P($2n$) describes the physics of the transition for relatively small values of K. This situation is quite similar in nature to that of the cubic problem in the context of continuous spins near $d = 4$, as mentioned in the introduction. The value of n_x will determine if the three-components cubic model, with small values of K, will be characterized by a '$2n$-Potts' or by a 'new cubic' critical behaviour.

In table 1 we list a few numerical results for the fixed points and the correlation length exponent ν resulting from the largest eigenvalue of the linearized recursion relations (26) near each fixed point. We emphasize again the crude approximation in our recursion relations. For example, the exact value of $\exp(\beta K_P(n))$ is $1+n^{1/2}$ (e.g. Mittag and Stephen 1971), and the values in table 1 clearly overestimate $\beta K_P(n)$.

Table 1. Fixed point parameters and critical exponent ν for the n-state Potts fixed point P(n) and for the 'new cubic' fixed point C.

n	P(n)		C		
	$(\beta K)^*$	ν	$(\beta J)^*$	$(\beta K)^*$	ν
0	$\sqrt{2n}$	∞	$n/2$	$-n^2/16$	$\frac{1}{2}$
1	0·96	0·82		Ising	
2	Ising			two decoupled Ising	
3	1·39	0·60	1·24	0·33	0·54
4	1·51	0·56	1·20	0·57	0·50
6	1·70	0·51	1·30	0·73	0·45
7	1·77	0·49	1·12	1·00	0·44
8	1·84	0·48	1·10	1·10	0·43
∞	$\sim\frac{2}{3}\ln n$	0·25	$\frac{1}{2}\ln 3$	$\sim\frac{2}{3}\ln n$	0·24

Similarly, we find too small a value for ν, or too large a value of α, compared to the series values (e.g. Kim and Joseph 1975 and references therein). The errors are similar to those encountered by Barber (1975) for the Ising case. One should therefore not take the actual numbers too seriously. However, it is reasonable to expect that some general tendencies will be maintained in better approximations. We wish to point out one such tendency, namely to the n-dependence of the exponent ν. For the n-state Potts model fixed point, P(n), the exponent ν decreases monotonically from $\nu = \infty$ in the limit $n \to 0$ to $\nu = \frac{1}{4}$ in the limit $n \to \infty$. Clearly, values of ν smaller than $1/d$ are not reasonable, as they imply values of the specific heat exponent α which are larger than 1 (using the scaling relation $d\nu = 2-\alpha$). These, in turn, imply a divergence of the energy as $T \to T_c$. Something special should therefore happen when $\nu = 1/d$, or when the largest eigenvalue becomes equal to b^d (in our case $b = \sqrt{2}$, $d = 2$). In a recent paper, Nienhuis and Nauenberg (1975) argue that a magnetic field eigenvalue b^d should be interpreted as representing a discontinuity in the magnetization, i.e. a first-order transition. One is therefore tempted to interpret a temperature eigenvalue b^d as representing a discontinuity in the internal energy, i.e. a *latent heat*, or in the quadrupolar order parameter $\langle S_i^2 \rangle$. Within our approximation, ν of P(n) equals $\frac{1}{2}$ for n between 6 and 7, and becomes 'unphysical' (less than $\frac{1}{2}$) for higher values of n. Could this correspond to the first-order transition expected for the Potts model for $n > 4$ (Baxter 1973)? We prefer to leave this as a question, until further work with better recursion relations is done.

An alternative explanation for the small values of ν found for large n is based on high-temperature series information. At $d = 3$, Kim and Joseph (1975) find that for all

$n > 2$, the transition is first-order (see however Straley 1974, Enting 1974). However, the magnetic susceptibility seems to diverge at T_c even though the transition is not continuous, with an exponent γ which is smaller than unity. Since in the recursion relation which we use the exponent η is exactly zero (Barber 1975), this implies that the exponent ν must be smaller than $\frac{1}{2}$ (using $\gamma = (2-\eta)\nu$). It is thus possible that improved relations will give $\gamma < 1$, but $\nu \geqslant \frac{1}{2}$. It is also possible that the singularities implied by the existence of a fixed point with large n are related to those found in the series by Kim and Joseph (1975).

A similar decrease of ν for large n occurs for the 'new cubic' fixed point. The interpretation here must probably be searched along similar directions as for the Potts case. In fact, ν becomes smaller than $\frac{1}{2}$ already at $n > 4$. Not much is known about the behaviour of the cubic model at $d = 2$. However, the series results at $d = 3$ do indicate a first-order transition for large n (Kim and Levy 1975). Mean field theory predicts a first-order transition for $n > 3$.

6. Discussion and summary

Although we treated the model exactly only near one dimension, and very approximately at two dimensions, there are some features which appear in all these dimensions and which should probably be considered as quite general. It is probably true that there will always be competition between four types of critical behaviour, i.e. that of the Ising model ($K \to \infty$), that of the n-state Potts model ($J \to 0$), that of the $2n$-state Potts model ($K \simeq J$) and that of the 'new cubic' fixed points. At $d = 1 + \epsilon$, all are stable at T_c except that of the $2n$-state Potts model, which is also a branch point. The same happens at $d = 2$, for $n < n_x$, and it is probably safe to conjecture that it will remain true for $d > 2$, with n_x depending on d (n_x is infinite at $d = 1 + \epsilon$, and of order 10 for $d = 2$). For $n > n_x$, the 'new cubic' fixed point becomes the branch point, which is unstable at T_c. For large n we find at $d = 2$ a peculiar decrease in the exponent ν, both for the 'new cubic' and for the n-state Potts fixed points. This may be associated with the transition becoming first order.

Once we have demonstrated the relation of the cubic model to the n-state or the $2n$-state Potts model, we can use independent information available for these models to interpret some of its critical properties. It would be very interesting to compare high-temperature series, with large values of n, for all these models. It seems that the first-order nature of the transition in the cubic model is closely related to that of the Potts model.

It is interesting to note that some systems which may be described by the cubic model do exhibit relatively high values of the exponent α (Kim *et al* 1975). This might be simply due to the fast increase in α (decrease in ν) as a function of n for the 'new cubic' fixed point, and not due to any tricritical behaviour.

Our main purpose here was to touch upon general features of the model, to exhibit the rich structure it has under the renormalization group and to raise a few conjectures concerning some explanations for these features. It would be highly desirable now to use more elaborate and systematic discrete spin renormalization group transformations for checking these conjectures. In particular, the relevance of the $2n$-state Potts model for cubic systems with $n > n_x$ and the appearance of the first-order transition via a temperature eigenvalue equal to b^d should be checked in detail. These checks are left for the future.

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